*between $\pi$ and $\pi'$ is equal in volume to the sum of the parts of $S$ and $S_1$ included between $\pi$ and $\pi'$.*

*Note.* In communicating the above question Mr. Elbert O. Brower, of Cicero, Illinois, says (in substance): "The truth of this proposition so easily follows from a consideration of the ordinary formula for getting the volume of a spherical segment, that it is difficult to suppose that it is unknown; yet I am led to wonder how it happens that, if known, it has not been accorded the prominence which it would seem to deserve. In sending it to the MONTHLY I wish to determine whether or not it is original with myself."—U. G. M.

## DISCUSSIONS.

### I. RELATING TO THE TRANSITION CURVE.

By GEORGE PAASWELL, Civil Engineer, New York City.

The transition curve is a so-called railroad spiral used to ease the approach to a circular curve. It is defined by its intrinsic equation $d\varphi/ds = 2ks$, where $s$ is the distance along the curve measured from the point of zero curvature and $k$ is a constant determined by the special data of the circular curve for which the transition is an easement. The integration of this equation gives $\varphi = ks^2$. Taking the length of the transition as $L$ and the radius of the circular curve as $R$, the value of $k$ is found from the intrinsic equation to be $1/2RL$.

From the geometry of the infinitesimal triangle $dy = ds \sin\varphi$ and $dx = ds \cos\varphi$, or, substituting from the above the values of $ds$ and $s$, we have

$$dy = \frac{1}{2\sqrt{k}}\frac{\sin\varphi}{\sqrt{\varphi}}\,d\varphi; \quad dx = \frac{1}{2\sqrt{k}}\frac{\cos\varphi}{\sqrt{\varphi}}\,d\varphi; \quad y = \frac{1}{2\sqrt{k}}\int_0^{\phi}\frac{\sin\varphi}{\sqrt{\varphi}}\,d\varphi\ ;$$

$$x = \frac{1}{2\sqrt{k}}\int_0^{\phi}\frac{\cos\varphi}{\sqrt{\varphi}}\,d\varphi.$$

Expanding the integrals, integrating term by term and replacing $k$ by its value $\sqrt{\varphi}/s$, we get

$$y = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(4n+3)\,\underline{|2n+1}}; \qquad x = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n}}{(4n+1)\,\underline{|2n}}.$$

Defining two functions,

$$\text{tran}\ \varphi = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n+1}}{(4n+3)\,\underline{|2n+1}} \quad \text{and cotran}\ \varphi = s\sum_0^{\infty}\frac{(-1)^n\varphi^{2n}}{(4n+1)\,\underline{|2n}},$$

we have $y = s$ tran $\varphi$, $x = s$ cotran $\varphi$, and, maintaining the same analogy, tatran $\varphi$ = tran $\varphi$/cotran $\varphi$, so that $y = x$ tatran $\varphi$. It may be advantageous to make up tables of these functions, similar to the trigonometric tables, and problems in the transition may be expressed in terms of these new functions. At present there is no rigorous mathematical discussion of this curve.

It may be interesting to establish the coördinates of the terminus of the transition. Since

$$\int_0^\infty \frac{\sin\varphi}{\sqrt{\varphi}}\,d\varphi = \int_0^\infty \frac{\cos\varphi}{\sqrt{\varphi}}\,d\varphi = \sqrt{\frac{\pi}{2}},$$

the coördinates become $x_\infty = y_\infty = \frac{1}{2}\sqrt{\pi/2k}$, $x = y = \frac{1}{2}\sqrt{\pi RL}$.

The coördinates of the transition are related to the Bessel functions as follows:

$$J_{1/2}(\varphi) = \frac{1}{\sqrt{\pi}}\frac{\sin\varphi}{\sqrt{\varphi}};\ J_{-1/2} = \frac{1}{\sqrt{\pi}}\frac{\cos\varphi}{\sqrt{\varphi}};\ C = \frac{1}{2}\int_0^\phi J_{-1/2}(\varphi)d\varphi;\ S = \frac{1}{2}\int_0^\phi J_{1/2}(\varphi)d\varphi;$$

whence $y = \sqrt{\pi RL}\,S(\varphi)$ and $x = \sqrt{\pi RL}\,C(\varphi)$. (Cf. Jahnke and Emde's Tables, p. 23 seq.)

## II. Relating to the Graph of a Cubic Equation having Complex Roots.

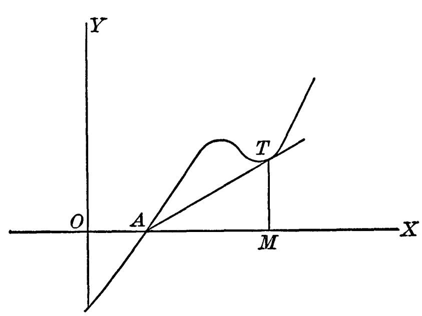
By Edwin S. Crawley, University of Pennsylvania.

The note on "The Graphical Solution of a Cubic Equation having Complex Roots," pp. 70–71 of the Monthly for February, 1918, recalls to my mind something similar which I learned a number of years ago and which might possibly interest some readers of the Monthly.

Every cubic with one real and two imaginary roots is expressible in the form $(x - k)(x^2 - 2px + p^2 + q^2) = 0$, and the graph of

$$y = (x - k)(x^2 - 2px + p^2 + q^2)$$

(*i. e.*, of $y = a_0x^3 + a_1x^2 + a_2x + a_3$) always has a form more or less like the figure. Then it is easy to show that $OM = p$ and $\tan MAT = q^2$, where $p \pm qi$




are the imaginary roots. $AT$ is the tangent to the curve drawn from its real intersection with $OX$.

For, if $OA = k$ the line $y = \lambda(x - k)$ will be tangent to the curve if

$$x^2 - 2px + p^2 + q^2 - \lambda = 0$$

has equal roots, that is, if $\lambda = q^2$; and $y = q^2(x - k)$ touches the curve at $x = p$.